Bloch martingales and conformal maps.

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Law of Iterated Logarithm (LIL) for Bloch functions.

$\exists\, c>0:\ \forall g\in\mathcal{B}$ a.e. $\zeta\in S^1$:

$$\varlimsup_{r\to 1-}\frac{|g(r\zeta)|}{\sqrt{\log\frac{1}{1-r}\,\log\log\log\frac{1}{1-r}}}\le c\|g\|_{\mathcal{B}}.$$

For a Bloch function g, define $G(z):=\int_0^z g(w)\,dw$.

Lemma. $G\in C_A(\overline{\mathbb{D}})$, - disk algebra satisfies Zygmund condition

$$|G(e^{i(\theta+t)})+G(e^{i(\theta-t)})-2G(e^{i\theta})|\le C\,t\,\|g\|_{\mathcal{B}}.$$

Remark. For $G\in C_A(\overline{\mathbb{D}})$, $G'\in\mathcal{B}\Leftrightarrow G$ is Zygmund-smooth.

Pf of Lemma. $|G(se^{i\theta})-G(re^{i\theta})|=\left|\int_r^s g(te^{i\theta})\,dt\right|\le$

$|g(re^{i\theta})|(s-r)+\frac{\|g\|_{\mathcal{B}}}{2}\int_r^s\log\left(\frac{1+t}{1-t}\right)dt$ $\quad\left(|g(te^{i\theta})|=\left|\int_r^t g'(le^{i\theta})\,dl+g(re^{i\theta})\right|\right.$

$\left.\le|g(re^{i\theta})|+\int_r^t\frac{1}{1-l^2}\,dl\,\|g\|_{\mathcal{B}}\le|g(re^{i\theta})|+\frac12\|g\|_{\mathcal{B}}\log\left(\frac{1+t}{1-t}\right)\right).$

Since $\int^1\log\left(\frac{1+t}{1-t}\right)<\infty$, $\exists\lim_{s\to1-}G(se^{i\theta})=:G(e^{i\theta})$.

Moreover, for $r=1-\frac{t}{\pi}$,

$$|G(e^{i(\theta+t)})+G(e^{i(\theta-t)})-2G(e^{i\theta})|\le\overbrace{\left|\int_r^1 g(se^{i(\theta+t)})+g(se^{i(\theta-t)})-2g(se^{i\theta})\,ds\right|}^{I}+\underbrace{|G(re^{i(\theta+t)})+G(re^{i(\theta-t)})-2G(re^{i\theta})|}_{II}$$

$$I\le\frac32\int_r^1\log\left(\frac{1+t}{1-t}\right)dt\,\|g\|_{\mathcal{B}}+|g(re^{i(\theta+t)})+g(re^{i(\theta-t)})-2g(re^{i\theta})|$$

$$\le\underbrace{|g(re^{i(\theta+t)})+g(re^{i(\theta-t)})-2g(re^{i\theta})|}_{I'}+6\|g\|_{\mathcal{B}}\underbrace{(1-r)}_{=\frac{t}{\pi}}.$$

But $I'\le 2t\|g\|_{\mathcal{B}}/(1-r^2)\le\pi\|g\|_{\mathcal{B}}$.

Finally, $II\le|t|^2\sup_{\zeta\in S^1}|g'(r\zeta)|\le\frac{t^2\|g\|_{\mathcal{B}}}{1-r^2}\le t\pi\|g\|_{\mathcal{B}}$.

Define now, for a dyadic interval I,

$$g_I:=\lim_{r\to1-}\frac{1}{|I|}\int_I g(re^{i\theta})\,d\theta=\lim_{r\to1-}\frac{-i}{|I|}\int_I e^{-i\theta}\frac{dG(re^{i\theta})}{d\theta}\,d\theta=$$

(by parts) $\dfrac{-ie^{-ib}G(e^{ib})+ie^{-ia}G(e^{ia})}{|I|}+\dfrac{1}{|I|}\int_I e^{-i\theta}G(e^{i\theta})\,d\theta$,

well-defined.

Define a function g_n by

$$g_n(\zeta)=\sum_{I\text{ - dyadic }n\text{-th generation}}g_I\,\chi_I(\zeta)$$

Observe that if I, J - adjacent, then $g_{I\cup J}=\frac12(g_I+g_J)$.

In particular, $E(g_{n+1}\mid\mathcal{M}_n)=g_n$, where $\mathcal{M}_n$ - dyadic partition of n-th level, the $\mathbb{P}$ is normalized linear measure on S^1.

So g_n is a martingale. But not just any martingale.

Def. A dyadic martingale is called Bloch if it has bounded jumps: $\exists C: \forall I, J$ - adjacent, $|f_n|_I - f_n|_J| \le C$.

Lemma $|g_n(\zeta) - g((1-2^{-n})\zeta)| \le C\|g\|_{\mathcal{B}}$ for some absolute constant C.

Pf. By the integration by parts formula for $g_{I,n}$ everything reduces to comparing $\frac{1}{|I|}\int_I g((1-2^{-n})e^{i\theta})d\theta$ and $g((1-2^{-n})\zeta)$ which is bounded, since hyperbolic diameter of $(1-2^{-n})I$ is bounded. ∎

Corollary g_n is a Bloch martingale.

Pf. $|g_n|I - g_n|J| \le |g_I - g((1-2^{-n})\zeta)| + |g_J - g((1-2^{-n})\zeta)| \le 2C\|g\|_{\mathcal{B}}$, where ζ is the common end of I and J. ∎

Pf of LIL.

Let $g \in \mathcal{B}$. Normalize: $\|g\|_{\mathcal{B}} = 1$. Take $f_n = \operatorname{Re} g_n$

Then $S_n \le Cn$, since $(f_{n+1} - f_n)^2 = (\frac{1}{2}(\underbrace{f_{I_\ell} - f_{I_r}}))^2 \le C$.

$I = I_\ell \cup I_r$ - decomposition of a dyadic interval of n-th generation.

Thus, if $S < \infty$, $\|f_n\|$ is bounded, by Levy.

if $S = \infty$, $|f_n| \le 2\sqrt{S_n \log\log S_n} \le C\sqrt{n \log\log n}$ for large n.

Then, by Lemma,

$|\operatorname{Re} g((1-2^{-n})e^{i\theta}) - f_n(\theta)| \le C_1$, and, by Lipschitz,

$|\operatorname{Re} g((1-2^{-n})e^{i\theta}) - \operatorname{Re} g(re^{i\theta})| \le C_2$, for $1-2^{-n} \le r < 1-2^{-n+1}$.

Thus

$|\operatorname{Re} g(re^{i\theta})| \le C_3\sqrt{\log\frac{1}{1-r}\log\log\log\frac{1}{1-r}}$ a.s.

Now do the same with $\operatorname{Im} g(re^{i\theta})$ ∎

Remark. All of this is sharp: every real Bloch martingale is generated by the real part of a Bloch function. When the Bloch norm is small, Nehari's criterion implies that $f = \log\varphi'$ for some φ - conformal. This allows us to prove that the lower bound in LIL can be achieved, with different C.

Corollary (Makarov's LIL): $\exists c > 0$ - absolute: $\forall f: \mathbb{D} \to \mathbb{C}$ - conformal.

A.e. $\zeta \in \mathbb{T}$: $\varlimsup_{r\to 1-} \frac{|\log f'(r\zeta)|}{\sqrt{\log\frac{1}{1-r}\log\log\log\frac{1}{1-r}}} \le c$.

Pf. $\log f' \in \mathcal{B}$, $\|\log f'\|_{\mathcal{B}} \le 6$ ∎

so, intuitively, a.e. $\zeta \in \mathbb{T}$, $|f'(r\zeta)|$ growth slower then any power of $\frac{1}{1-r}$.

It is the key step in the proof of Makarov's dimension Thm.

Another proof of Makarov's LIL (w/o probability!)

Thm (Integral Means Estimate)

$g \in \mathcal{B}$, $g(0) = 0$. Then

$\frac{1}{...}\int |g(r\zeta)|^{2n} |d\zeta| \le n!\, \|g\|^{2n}_{...} \left(\log\frac{1}{...}\right)^n$

$g\in\mathcal{B}$, $g(0)=0$. Then

$$\frac{1}{2\pi}\int_{\mathbb{T}} |g(r\zeta)|^{2n}\,|d\zeta| \le n!\,\|g\|_{\mathcal{B}}^{2n}\left(\log\frac{1}{1-r^2}\right)^n.$$

Pf. Induction by n. $n=0$ — trivial.

Observe, that for $g\in A(\mathbb{D})$, $z=re^{i\theta}$, we have

$$\left(r\frac{\partial}{\partial r}\right)^2|g(z)|^p+\left(\frac{\partial}{\partial\theta}\right)^2|g(z)|^p=p^2|g(z)|^p\left|z\frac{g'(z)}{g(z)}\right|^2.$$

Integrate $0\le\theta\le 2\pi$. Second term on RHS disappears, and we get Hardy's identity:

$$\boxed{\frac{d}{dr}\left(r\frac{d}{dr}\int_0^{2\pi}|g(re^{i\theta})|^p\,d\theta\right)=p^2\,r\int_0^{2\pi}|g(re^{i\theta})|^{p-2}|g'(re^{i\theta})|^2\,d\theta}$$

Use it: assume Thm proved for n.

Then

$$\frac{d}{dr}\left(r\frac{d}{dr}\right)\left(\frac{1}{2\pi}\int_{\mathbb{T}}|g(r\zeta)|^{2n+2}|d\zeta|\right)=\frac{4(n+1)^2}{2\pi}\,r\int_{\mathbb{T}}|g(r\zeta)|^{2n}|g'(r\zeta)|^2|d\zeta|$$

But $|g'(r\zeta)|\le\dfrac{\|g\|_{\mathcal{B}}}{1-r^2}$, so, by induction,

$$\le 4(n+1)^2\,r\,n!\left(\log\frac{1}{1-r^2}\right)^n\|g\|_{\mathcal{B}}^{2n}(1-r^2)^{-2}\|g\|_{\mathcal{B}}^2\le$$

$$(n+1)!\,\|g\|_{\mathcal{B}}^{2n+2}\frac{d}{dr}\left(r\frac{d}{dr}\left(\log\frac{1}{1-r^2}\right)^{n+1}\right),$$ and we get our statement.

Proof of Makarov LIL. $\|g\|_{\mathcal{B}}=1$, to simplify notations.

$g^*(s,\zeta):=\max\limits_{0\le r\le 1-e^{-s}}|g(r\zeta)|$ — Hardy-Littlewood Maximal function.

Hardy-Littlewood Thm:

$$\int_{\mathbb{T}} g^*(s,\zeta)^{2n}|d\zeta|\le\frac{k}{2\pi}\int_{\mathbb{T}}\left|g((1-e^{-s})\zeta)\right|^{2n}|d\zeta|\overset{\text{previous}}{\le}k\,n!\,s^n.$$

Define: $\varphi_n(s):=-n\dfrac{d}{ds}\left((\log s)^{-1/n}\right)=s^{-1}(\log s)^{-1-\frac{1}{n}}$.

Multiply by $s^{-n}\varphi_n(s)$, integrate:

$$\int_{\mathbb{T}}\left(\int_e^\infty g^*(s,\zeta)^{2n}s^{-n}\varphi_n(s)\,ds\right)|d\zeta|\le k\,n!\int_e^\infty\varphi_n(s)\,ds=$$

$k\,n!\,n$.

so, by Chebyshev, $\exists A_n\subset\mathbb{T}$, $|A_n|>2\pi-\frac{k}{n^2}$, such that

$$\int_e^\infty g^*(s,\zeta)^{2n}s^{-n}\varphi_n(s)\,ds\le n!\,n^3 \text{ for } \zeta\in A_n.$$

Observe: $-\dfrac{d}{ds}\left(s^{-n}(\log s)^{-1-\frac{1}{n}}\right)\le 3ns^{-n}\varphi_n(s)$, so

for $\zeta\in A_n$, $e\le\sigma<\infty$, we have

$$g^*(\sigma,\zeta)^{2n}\sigma^{-n}(\log\sigma)^{-1-1/n}\le 3n\int_\sigma^\infty g^*(s,\zeta)^{2n}s^{-n}\varphi_n(s)\,ds\le$$

$3\,n!\,n^4$.

Observe that the lower limit of A_n, $A=\bigcup_{k=1}^\infty\bigcap_{n=k}^\infty A_n$ has full measure, since $\sum|A_n|<\infty$ (Borel-Cantelli).

If $\zeta\in A$, then $\exists k$: $\zeta\in A_n$ $\forall n\ge k$.

Let $r<1$, such that $n := \lfloor \log\log\sigma \rfloor \geq k$, $\sigma = \log\frac{1}{1-r}$.
Since $\log\log\sigma \geq n$, $\log\sigma \leq e^{n+1}$, we have

$$\frac{|g(r\xi)|^2}{\sigma\log\log\sigma} \leq \frac{g^*(\sigma,\xi)^2}{\sigma\log\log\sigma} \leq \left(\frac{3\,n!\,n^4}{n^n}\right)^{1/n} e^{(n+1)^2/n^2} \xrightarrow[n\to\infty]{} 1, \text{ by}$$

Stirling ∎